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NILPOTENT INJECTORS IN ALTERNATING GROUPS

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ABSTRACT

An N-Injector in an arbitrary finite group is defined as a maximal nilpotent subgroup of G containing a subgroup A of G of maximal order, satisfying class$(A) \leqq 2$. In a previous paper the N-Injectors of Sym(n) were determined. In this paper we determine the N-Injectors of Alt(n), after having determined the set of all nilpotent subgroups, A, of Sym(n) of maximal order satisfying class$(A) \leqq 2$. It is shown that the set of N-Injectors of Alt(n) consists of a unique conjugacy class, and if $n \neq 9$, it coincides with the set of the nilpotent subgroups of Alt(n) of maximal order.

A. Introduction

In a previous paper [1] the author has defined N-Injectors in an arbitrary finite group G, and among other results determined the N-Injectors of Sym(n). In the present paper we will continue [1] and deal with N-Injectors of Alt(n), as well as with some generalizations.

In Sections B and C preliminary results are derived. In Section D the N-Injectors of Alt(n) are determined, and they are shown to consist of a unique conjugacy class. It will also be proved that if $n \neq 9$, then the N-Injectors of Alt(n) are the nilpotent subgroups of maximal order. We will conclude the paper by proving that if $G = \text{Alt}(n)$, then conjecture 1 and conjecture 2 of [1] hold. To begin, we will review some definitions and notation from [1].

Definitions and Notation. Let π be a set of primes and G be a finite group.

(a) $d(2,G)$ ($d(\pi,2,G)$) will denote the maximum of orders of all nilpotent subgroups (nilpotent π-subgroups) of G of class at most two.

(b) $\mathscr{A}(2,G)$ ($\mathscr{A}(\pi,2,G)$) will denote the set of all nilpotent subgroups (nilpotent π-subgroups) of G of class at most two, having order $d(2,G)$ ($d(\pi,2,G)$).

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(c) The N-Injectors (π-N-Injectors) of G are maximal nilpotent subgroups (maximal nilpotent π-subgroups) of G containing an element of $\mathcal{A}(2, G)$ ($\mathcal{A}(\pi, 2, G)$). The set of N-Injectors (π-N-Injectors) of G will be denoted by $NI(G)$ ($NI(\pi, G)$).

(d) $d(\infty, G)$ ($d(\pi, \infty, G)$) will denote the order of nilpotent subgroups (nilpotent π-subgroups) of G of maximal order.

(e) $\mathcal{A}(\infty, G)$ ($\mathcal{A}(\pi, \infty, G)$) will denote the set of all nilpotent subgroups (nilpotent π-subgroups) of G of maximal order.

In [1] these definitions were discussed in detail, and the following two conjectures were stated.

CONJECTURE 1. If G is a finite group and π a set of primes, then $NI(\pi, G)$ is a conjugacy class.

CONJECTURE 2. If G is a finite group and π a set of primes, then $\mathcal{A}(\pi, \infty, G)$ is a conjugacy class.

In [1] it was proved that if $G = \mathrm{Sym}(n)$, then conjecture 1 and conjecture 2 hold. As mentioned above, in this paper we confirm conjectures 1 and 2 for the alternating groups.

B. Evaluation of $d(2, S_2(\mathrm{Alt}(2n)))$

In this section we will first determine $\mathcal{A}(2, S_2(\mathrm{GL}(n, 3)))$ and then $\mathcal{A}(2, S_2(\mathrm{Sym}(2n)))$. As a consequence $d(2, S_2(\mathrm{Alt}(2n)))$ will be evaluated. It was already proved in [1, sec. B] that $S_2(\mathrm{Sym}(2n))$ can be embedded in $S_2(\mathrm{GL}(n, 3))$ and that

$$d(2, S_2(\mathrm{GL}(n, 3))) = d(2, S_2(\mathrm{Sym}(2n))) = 2^{n \cdot [n/2]}.$$

We will use these facts freely in the sequel.

LEMMA B.1. *Let a p-group P act faithfully and irreducibly on a vector space V of dimension n over* $\mathrm{GF}(q)$. *Suppose that P is not cyclic, generalized quaternion, dihedral or semi-dihedral, then:*

(a) *P has a subgroup H of index p such that we can write V as a direct sum $V = V_1 + V_2 + \cdots V_p$, where each V_i is an H invariant subspace.*

(b) *Let $K_i = C_H(V_i)$; then $K_i \neq 1$ for $1 \leqq i \leqq p$ and if $x \in P \setminus H$, then $V_i x = V_{i'}$, where the permutation $i \to i'$ is a p-cycle.*

(c) *If in addition* $\mathrm{class}(P) \leqq 2$, *then $|K_i| \leqq p$ for $1 \leqq i \leqq p$.*

PROOF. Parts (a) and (b) are a theorem of Roquette [3, 19.2]. We will prove part (c). Take $x \in P \setminus H$ and define a mapping:

$$\varphi : K_1 \to \Omega_1(Z(P)) \quad \text{by} \quad \varphi(y) = [y, x] \ \forall y \in K_1.$$

We will show that φ is a well defined monomorphism. Since P is of class at most two $\varphi(y) \in P' \subseteq Z(P)$. Since for each i, $1 \leqq i \leqq p$, $K_i = K_1^{x^j}$ for some j, $1 \leqq j \leqq p$, it follows that $C_{K_1}(x)$ acts trivially on V; hence $C_{K_1}(x) = 1$. This implies that:

$$[H, K_1] \subseteq K_1 \cap P' \subseteq K_1 \cap Z(P) = 1 \quad \text{and} \quad K_1 \subseteq Z(H).$$

Clearly, for each $y \in K_1$, $y^x = y[y, x]$ and by induction $y^{x^i} = y[y, x]^i$. Since $x^p \in H$ and $K_1 \subseteq Z(H)$, $y = y^{x^p} = y[y, x]^p$, and we obtain that $[y, x]^p = 1$. Thus, $\varphi(y) \in \Omega_1(Z(P))$ and φ is well defined. Moreover, as

$$\varphi(y_1 y_2) = [y_1 y_2, x] = [y_1, x]^{y_2}[y_2, x] = [y_1, x][y_2, x] = \varphi(y_1)\varphi(y_2)$$

φ is a homomorphism, and since $\ker \varphi = C_{K_1}(x) = 1$, φ is a monomorphism.

P acts irreducibly on V so by [3, 15.4] $Z(P)$ is cyclic. Therefore, $|\Omega_1(Z(P))| = p$, which implies that $|\varphi(K_1)| \leqq p$; hence $|K_1| \leqq p$. Now the proof of (c) is complete.

Lemma B.2. *If P is a 2-subgroup of* $\mathrm{GL}(n, 3)$ *of class at most two and of order* $2^{n+[n/2]}$, *which acts irreducibly, then either* (a) *or* (b) *occurs*:

(a) $n = 1$ *and* $|P| = 2$;

(b) $n = 2$, $|P| = 8$ *and P is one of the following*: *cyclic, quaternion or dihedral.*

Proof. We will prove first that $n \leqq 2$. If P is none of the following groups, cyclic, generalized, quaternion, dihedral or semi-dihedral, then apply Lemma B.1. It follows that $n = 2t$ and there are two subgroups of P, K_1 and H, such that $K_1 \subseteq H$, $|K_1| \leqq 2$, $|P : H| = 2$, $H/K_1 \subseteq S_2(\mathrm{GL}(t, 3))$ and $\mathrm{class}(H/K_1) \leqq 2$. Consequently, $|H : K_1| \leqq 2^{t+[t/2]}$ and hence

$$2^{3t} = |P| = |P : H|\,|H : K_1|\,|K_1| \leqq 2^{t+[t/2]+2},$$

yielding $t \leqq 1$ and hence $n \leqq 2$. As $\mathrm{class}(P) \leqq 2$, P is not semi-dihedral. If P is generalized quaternion or dihedral, then $\mathrm{class}(P) \leqq 2$ implies that $|P| = 8$, and in view of the order of P it follows that $n = 2$. If P is cyclic, then by [1, p. 266] $|P| \leqq 2^3[n/2] + 2$ and the order of P implies that $n \leqq 3$. But as $S_2(\mathrm{GL}(3, 3))$ does not contain a cyclic subgroup of order 16, we obtain in this case $n \leqq 2$ as well. We have proved that in any case $n \leqq 2$. As $S_2(\mathrm{GL}(2, 3))$ is semi-dihedral of order 16 and class three, [2, p. 191] implies the structure of P in the case $n = 2$, and Lemma B.2 is proved.

Lemma B.3. *If P is a 2-subgroup of* $\mathrm{GL}(n, 3)$ *of class at most two and of order* $2^{n+[n/2]}$, *then*:

(a) *P is a direct product of its projections on its irreducible subspaces.*

(b) *If n is even, then the dimension of each irreducible subspace is two.*

(c) *If n is odd, then the dimension of one irreducible subspace is one, while that of the other subspaces is two.*

PROOF. Consider $\Pi_{i=1}^{s} P_i$, the direct product of the projections of P on its irreducible subspaces V_i. It is a subgroup of $\mathrm{GL}(n,3)$ of class at most two, hence of order $2^{n+[n/2]}$ at most. But as $|P| = 2^{n+[n/2]}$, it follows that $P = \Pi_{i=1}^{s} P_i$ and (a) is proved. Let $\dim(V_i) = n_i$; by Lemma B.2, n_i is either 1 or 2. Now the order of P implies (b) and (c).

Combining Lemma B.2 and Lemma B.3, the structure of the elements of the set $\mathcal{A}(2, S_2(\mathrm{GL}(n,3)))$ is clear.

COROLLARY B.4. *Let $P \in \mathcal{A}(2, S_2(\mathrm{GL}(n,3)))$. Then P is the direct product $P = \Pi_{i=1}^{n/2} D_i$ if n is even and $P = \Pi_{i=0}^{[n/2]} D_i$ if n is odd, where D_0 is a group of order 2 and the D_i's for $1 \leqq i \leqq [n/2]$ are dihedral, quaternion or cyclic of order 8.*

LEMMA B.5. *Let P and D_i be as in Corollary B.4 and suppose that P has a faithful permutation representation on Ω of degree 2n. Then the following hold:*

(a) *The representation of P is a direct product of the representations of the D_i's, $0 \leqq i \leqq [n/2]$.*

(b) *For $1 \leqq i \leqq [n/2]$, D_i is the dihedral group and it is represented on 4 symbols.*

(c) *D_0 is represented on 2 symbols.*

PROOF. By induction on n. Clearly the Lemma holds for $n = 1$ and 2. Assume $n > 2$, and let $\Omega_1, \Omega_2, \cdots, \Omega_k$ be a partition of Ω into transitive orbits of P. Consider $\Pi_{i=1}^{k} P_i$, the direct product of the projections of P on the Ω_i's, $1 \leqq i \leqq k$. This is a group of class at most 2 and P can be embedded in it. Now the order of P implies that $P = \Pi_{i=1}^{k} P_i$ and $|P_i| = 2^{n_i+[n_i/2]}$, where $2n_i = |\Omega_i|$ is a power of 2. In view of the order of P at most one Ω_i has cardinality 2, while all the others have cardinality divisible by 4. As P_i can be embedded in $S_2(\mathrm{GL}(n_i,3))$ and has order $2^{n_i+[n_i/2]}$, by Corollary B.4 it has the same form as P and an induction hypothesis can be applied if $k \geqq 2$. Thus the Lemma is proved in the case when P is not transitive. If P is transitive, the stabilizer of a point is a subgroup H having the property $\mathrm{core}(H) = 1$. As for $1 \leqq i \leqq s$, every subgroup of D_i of order 4 contains a central involution, the order of the projection of H on each D_i is at most 2 and hence $|H| \leqq 2^{[n/2]}$. It follows that $2^{n+[n/2]} = |P| = 2n\,|H| \leqq 2n 2^{[n/2]}$ which contradicts the assumption $n > 2$. The proof of Lemma B.5 is complete.

COROLLARY B.6. *If $P \in \mathscr{A}(2, S_2(\mathrm{Sym}(2n)))$, then P is a direct product of permutation groups as follows:*

(a) $P = \prod_{i=1}^{n/2} S_2(\mathrm{Sym}(4))$ *if n is even.*

(b) $P = (\prod_{i=1}^{[n/2]} S_2(\mathrm{Sym}(4))) \times \mathrm{Sym}(2)$ *if n is odd.*

PROOF. If $P \in \mathscr{A}(2, S_2(\mathrm{Sym}(n)))$, then P can be embedded in $S_2(\mathrm{GL}(n,3))$ and then $P \in \mathscr{A}(2, S_2(\mathrm{GL}(n,3)))$. Now apply Corollary B.4 and Lemma B.5.

COROLLARY B.7. $d(2, S_2(\mathrm{Alt}(2n))) = 2^{n+[n/2]-1}$.

PROOF. Clearly $d(2, S_2(\mathrm{Alt}(2))) = 1$. So suppose that $n > 1$. Then $d(2, S_2(\mathrm{Alt}(2n)))$ equals either $2^{n+[n/2]}$ or $2^{n+[n/2]-1}$. If $d(2, S_2(\mathrm{Alt}(2n))) = 2^{n+[n/2]}$, then $\mathscr{A}(2, S_2(\mathrm{Alt}(2n))) \subseteq \mathscr{A}(2, S_2(\mathrm{Sym}(2n)))$. But if $P \in \mathscr{A}(2, S_2(\mathrm{Sym}(2n)))$, then it follows by Corollary B.6 that P has odd permutations, for example a single transposition in an arbitrary $S_2(\mathrm{Sym}(4))$, a contradiction.

C. Arithmetical lemmas

We will introduce some notation. Let m be a positive integer and p a prime. Define $\varphi_2(1) = \varphi_\infty(1) = 1$ and

$$\varphi_2(m) = \begin{cases} 2 & \text{if } m = 2, \\ 2^{3\cdot 2^{\alpha-2}} & \text{if } m = 2^\alpha \text{ and } \alpha \geqq 2, \\ p^{p^{\alpha-1}} & \text{if } m = p^\alpha,\ p > 2 \text{ and } \alpha \geqq 1, \\ \prod_{i=1}^{s} \varphi_2(p_i^{\alpha_i}) & \text{if } m = \prod_{i=1}^{s} p_i^{\alpha_i}, \text{ where the } p_i\text{'s are} \\ & \quad \text{distinct primes for } 1 \leqq i \leqq s; \end{cases}$$

$$\varphi_\infty(m) = \begin{cases} p^{(p^\alpha-1)/(p-1)} & \text{if } m = p^\alpha, \\ \prod_{i=1}^{s} \varphi_\infty(p_i^{\alpha_i}) & \text{if } m = \prod_{i=1}^{s} p_i^{\alpha_i}, \text{ where the } p_i\text{'s are} \\ & \quad \text{distinct primes for } 1 \leqq i \leqq s. \end{cases}$$

Using our notation it follows that if M is a maximal nilpotent subgroup of $\mathrm{Sym}(n)$ corresponding to the partition $\{n_1, \cdots, n_r\}$ of n (see [1, p. 267] and [4, I.5]), then by [4, I.5] $|M| = \prod_{i=1}^{r} \varphi_\infty(n_i)$ and by [1, B.5] $d(2, M) = \prod_{i=1}^{r} \varphi_2(n_i)$.

LEMMA C.1. *If p is an odd prime and $p^\alpha = \sum_{i=1}^{s} 2^{\alpha_i}$ is the 2-adic representation of p^α, then:*

(a) $\varphi_2(p^\alpha) < \frac{1}{2}\prod_{i=1}^{s} \varphi_2(2^{\alpha_i})$ *if* $p^\alpha \notin \{3,5\}$,

(b) $\varphi_\infty(p^\alpha) < \frac{1}{2}\prod_{i=1}^{s} \varphi_\infty(2^{\alpha_i})$ *if* $p^\alpha \notin \{3,5,9\}$.

PROOF. (a) The inequality can be easily checked for $p^\alpha \leqq 27$, so assume $p^\alpha > 27$. If $p^\alpha \equiv 3 \pmod 4$, then $\prod_{i=1}^s \varphi_2(2^{\alpha_i}) = 2^{-5/4}\prod_{i=1}^s 2^{3\cdot 2^{\alpha_i-2}}$ and if $p^\alpha \equiv 1 \pmod 4$, then $\prod_{i=1}^s \varphi_2(2^{\alpha_i}) = 2^{-3/4}\prod_{i=1}^s 2^{3\cdot 2^{\alpha_i-2}-1}$. So in any case we get in view of $p^\alpha > 27$ and $p \geqq 3$,

$$\frac{1}{2}\prod_{i=1}^{s} \varphi_2(2^{\alpha_i}) \geqq 2^{-9/4}\prod_{i=1}^{s} 2^{3\cdot 2^{\alpha_i-2}} = 2^{3p^\alpha/4-9/4} > 2^{2p^\alpha/3} > p^{p^\alpha-1} = \varphi_2(p^\alpha).$$

(b) If $p^\alpha \neq 3$ is of the form $2^\beta - 1$, then by [3, 19.3] $\alpha = 1$ and the inequality follows easily. So assume that p is not of this form and hence $s < \log_2 p^\alpha$. Since

$$\tfrac{1}{2}\prod_{i=1}^{s} 2^{2^{\alpha_i-1}} = \tfrac{1}{2}2^{p^\alpha-s} > \tfrac{1}{2}2^{p^\alpha-\log_2 p^\alpha},$$

it suffices to prove that $p^{(p^\alpha-1)/(p-1)} < \frac{1}{2}2^{p^\alpha-\log_2 p^\alpha}$. We will prove the following equivalent inequality:

$$(*) \qquad 1 > \left(\frac{1}{p-1} + \frac{\alpha}{p^\alpha-1}\right)\log_2 p.$$

(i) If $p \geqq 7$, then $1 > (2/(p-1))\log_2 p$. Since $1/(p-1) \geqq \alpha/(p^\alpha-1)$, the inequality (*) holds for $p \geqq 7$.

(ii) If $p = 5$ and $\alpha \geqq 2$, then

$$\frac{\alpha}{5^\alpha-1} \leqq \frac{2}{5^2-1} \quad \text{and} \quad 1 > \left(\frac{1}{5-1} + \frac{2}{5^2-1}\right)\log_2 5,$$

yielding (*).

(iii) If $p = 3$ and $\alpha \geqq 3$, then

$$\frac{\alpha}{3^\alpha-1} \leqq \frac{3}{3^3-1} \quad \text{and} \quad 1 > \left(\frac{1}{3-1} + \frac{3}{3^3-1}\right)\log_2 3,$$

yielding (*).

LEMMA C.2. *If m is a positive integer, not a power of a prime, then there is a partition $\{n_1, \cdots, n_r\}$ of m into powers of a fixed prime which divides m such that*:

(a) $\varphi_2(m) < \frac{1}{2}\prod_{i=1}^r \varphi_2(n_i)$,

(b) $\varphi_\infty(m) < \frac{1}{2}\prod_{i=1}^r \varphi_\infty(n_i)$.

PROOF. Using induction it is easily seen that it suffices to prove the Lemma in the case $m = p^\alpha q^\beta$, where p and q are distinct primes and $p > q$. We will deal first with the case $m = 6$. Considering the partition $\{4, 2\}$ we obtain $\varphi_2(4)\varphi_2(2) = \varphi_\infty(4)\varphi_\infty(2) = 8\cdot 2 = 16$ while $\varphi_2(6) = \varphi_\infty(6) = 6$, and we are through.

Now we can assume that $p^\alpha \geqq 3$ and if $p^\alpha = 3$, then $q = 2$ and $\beta \geqq 2$. Consider the partition of m into q^β parts, where each part equals p^α. Parts (a) and (b) will be proved separately.

(a) (i) If $m = p^\alpha \cdot 2$, then $p^\alpha > 3$ implies that

$$\varphi_2(m) = 2p^{p^\alpha - 1} < \tfrac{1}{2}p^{2p^\alpha - 1} = \tfrac{1}{2}(\varphi_2(p^\alpha))^2$$

and this case is settled.

(ii) If $m = p^\alpha \cdot 2^\beta$ where $\beta \geqq 2$, then the desired inequality is equivalent to

$$2^{3/4} < (\tfrac{1}{2})^{1/2^\beta} p^{p^\alpha - 1 \cdot (2^\beta - 1)/2^\beta}.$$

But as $(\frac{1}{2})^{1/4} \leqq (\frac{1}{2})^{1/2^\beta}$ and $3/4 \leqq (2^\beta - 1)/2^\beta$, the last inequality follows and this case is settled too.

(iii) If $m = p^\alpha q^\beta$ where $q \geqq 3$, then the desired inequality is equivalent to

$$q^{1/q} < (\tfrac{1}{2})^{1/q^\beta} p^{p^\alpha - 1 \cdot (q^\beta - 1)/q^\beta}.$$

But as $(\frac{1}{2})^{1/3} \leqq (\frac{1}{2})^{1/q^\beta}$, $\frac{2}{3} \leqq (q^\beta - 1)/q^\beta$ and $q^{1/q} \leqq 3^{1/3}$ the last inequality follows and the proof of (a) is complete.

(b) The desired inequality is equivalent to

$$q^{1/(q-1)} < (\tfrac{1}{2})^{1/(q^\beta - 1)} p^{(p^\alpha - 1)/(p-1)}.$$

If $p^\alpha = 3$, then $q = 2$ and $\beta \geqq 2$ and the inequality follows from $(\frac{1}{2})^{1/3} \leqq (\frac{1}{2})^{1/(q^\beta - 1)}$ and $2 = q^{1/(q-1)}$. If $p^\alpha \geqq 5$, then the inequality follows as $\frac{1}{2} \leqq (\frac{1}{2})^{1/(q^\beta - 1)}$ and $q^{1/(q-1)} \leqq 2$. Now the proof of part (b) is complete.

D. The main theorems

THEOREM D.1. (a) *$\mathcal{A}(\infty, \mathrm{Alt}(5))$ is the set of 5-Sylow subgroups of* Alt(5).

(b) *If $n = 3, 6, 9$, then $\mathcal{A}(\infty, \mathrm{Alt}(n))$ is the set of 3-Sylow subgroups of* $\mathrm{Alt}(n)$.

(c) *If $n \not\equiv 3 \pmod 4$ and $n \neq 5, 6, 9$ then $\mathcal{A}(\infty, \mathrm{Alt}(n))$ is the set of 2-Sylow subgroups of* $\mathrm{Alt}(n)$.

(d) *If $n \equiv 3 \pmod 4$ and $n > 3$, then $\mathcal{A}(\infty, \mathrm{Alt}(n))$ is the set of all subgroups generated by any 3-cycle and a 2-Sylow subgroup of* $\mathrm{Alt}(n-3)$ *on the remaining $n - 3$ symbols.*

(e) *In any case $\mathcal{A}(\infty, \mathrm{Alt}(n))$ is a conjugacy class.*

PROOF. In view of the structure of maximal nilpotent subgroups of $\mathrm{Sym}(n)$ (see [4, I.5]) a simple checking yields (a) and (b). Let $M_A \in \mathcal{A}(\infty, \mathrm{Alt}(n))$ where $n \neq 3, 5, 6, 9$ and let $M_A \subseteq M$, where M is a maximal nilpotent subgroup of $\mathrm{Sym}(n)$, corresponding to the partition $\{n_1, n_2, \cdots, n_k\}$ of n. If one of the n_i's is not a power of a prime, then by applying Lemma C.2 (b) to that n_i it follows that there is a nilpotent subgroup of Sym(n), M' (not necessarily maximal) satisfying $|M'|>2|M|$, contradicting $M_A \in \mathscr{A}(\infty, \mathrm{Alt}(n))$. Hence each n_i is a power of a prime. Applying Lemma C.1 (b), it follows that each n_i is either a power of 2 or belongs to the set $\{3,5,9\}$. A simple argument yields:

(i) If 3 occurs twice in the partition, then $n=6$. (Otherwise, the partition is of one of the following types: $\{3,3,9,\cdots\}$, $\{3,3,5,\cdots\}$, $\{3,3,2^\alpha,\cdots\}$, $\alpha \geqq 1$, $\{3,3,1\}$, which can be replaced by $\{8,4,2,1,\cdots\}$, $\{8,2,1,\cdots\}$, $\{4,2,2^\alpha,\cdots\}$, $\{4,3\}$, respectively, the latter partitions corresponding to an M' satisfying $|M'|> 2|M|$.)

(ii) If 5 (9) belongs to the partition, then again by a suitable replacement argument it follows that $n=5$ ($n=9$).

As (i) and (ii) contradict our assumption $n \neq 3,5,6,9$, we can deduce that the n_i's are powers of 2 and possibly one of them is 3. Since M is a maximal nilpotent subgroup of Sym(n), the n_i's are either the terms in the 2-adic representation of n, or 3 occurs and the n_i's are the terms in the 2-adic representation of $n-3$. Using the information about $\mathscr{A}(\infty, \mathrm{Sym}(n))$ (see [1]), (c) and (d) follow. Clearly $\mathscr{A}(\infty, \mathrm{Alt}(n))$ is a conjugacy class in cases (a), (b) and (c). As Alt(n) is transitive on the $\binom{n}{3}$ 3-subsets of $\{1,2,\cdots,n\}$, it follows that $\mathscr{A}(\infty, \mathrm{Alt}(n))$ is a conjugacy class in case (d) as well. This completes the proof of (e) and hence of Theorem D.1.

Theorem D.2. (a) *NI*(Alt(5)) *is the set of* 5-*Sylow subgroups of* Alt(5).

(b) *If* $n=3,6$, *then* *NI*(Alt(n)) *is the set of* 3-*Sylow subgroups of* Alt(n).

(c) *If* $n \neq 3 \pmod 4$ *and* $n \neq 5,6$, *then* *NI*(Alt(n)) *is the set of* 2-*Sylow subgroups of* Alt(n).

(d) *If* $n=3 \pmod 4$ *and* $n>3$, *then* *NI*(Alt(n)) *is the set of all subgroups generated by any* 3-*cycle and a* 2-*Sylow subgroup of* Alt($n-3$) *on the remaining* $n-3$ *symbols.*

(e) *In any case,* *NI*(Alt(n)) *is a conjugacy class.*

Proof. Theorem D.1 implies (a) and (b). Let $M_A \in NI(\mathrm{Alt}(n))$ where $n \neq 3,5,6$ and let $M_A \subseteq M$, where M is a maximal nilpotent subgroup of Sym(n) corresponding to the partition $\{n_1, n_2, \cdots, n_k\}$ of n. If one of the n_i's is not a power of a prime, then by applying Lemma C.2(a) to that n_i it follows that there is a nilpotent subgroup of Sym(n), M' (not necessarily maximal) satisfying $d(2,M')>2d(2,M)$ contradicting $M_A \in NI(\mathrm{Alt}(n))$. Hence each n_i is a power of a prime. Applying Lemma C.1(a) it follows that each n_i is either a power of 2 or belongs to the set $\{3,5\}$. As in Theorem D.1 it follows that:

(i) If 3 occurs twice in the partition, then $n = 6$, contradicting our assumption.

(ii) If 5 belongs to the partition, then $n = 5$, contradicting our assumption again.

Now we can deduce that the n_i's are powers of 2 and possibly one of them is 3. Since M is a maximal nilpotent subgroup of $\mathrm{Sym}(n)$, the n_i's are either the terms in the 2-adic representation of n or 3 occurs and the n_i's are the terms in the 2-adic representation of $n-3$. Using the information about $NI(\mathrm{Sym}(n))$ (see [1]), (c) and (d) follow. The proof of (e) is similar to that of Theorem D.1(e).

THEOREM D.3. *Let π be a set of primes, then*:

(a) $NI(\pi, \mathrm{Alt}(n))$ *consists of a unique conjugacy class.*

(b) $\mathscr{A}(\pi, \infty, \mathrm{Alt}(n))$ *consists of a unique conjugacy class.*

PROOF. Parts (a) and (b) will be proved simultaneously. Assume first that $2 \in \pi$. The following table determines the sets $NI(\pi, \mathrm{Alt}(n))$ and $\mathscr{A}(\pi, \infty, \mathrm{Alt}(n))$ in the four possible subcases. We omit the proofs, which follow easily from Theorems D.1 and D.2 and the structure of maximal nilpotent subgroups of $\mathrm{Sym}(n)$, [4, I.5].

	$NI(\pi, \mathrm{Alt}(n))$	$\mathscr{A}(\pi, \infty, \mathrm{Alt}(n))$
$3 \in \pi$ $5 \in \pi$	$NI(\mathrm{Alt}(n))$	$\mathscr{A}(\infty, \mathrm{Alt}(n))$
$3 \in \pi$ $5 \notin \pi$	$NI(\mathrm{Alt}(n))$ if $n \neq 5$ $\mathrm{Syl}_2(\mathrm{Alt}(n))$ if $n = 5$	$\mathscr{A}(\infty, \mathrm{Alt}(n))$ if $n \neq 5$ $\mathrm{Syl}_2(\mathrm{Alt}(n))$ if $n = 5$
$3 \notin \pi$ $5 \in \pi$	$\mathrm{Syl}_2(\mathrm{Alt}(n))$ if $n \neq 5$ $\mathrm{Syl}_5(\mathrm{Alt}(n))$ if $n = 5$	$\mathrm{Syl}_2(\mathrm{Alt}(n))$ if $n \neq 5$ $\mathrm{Syl}_5(\mathrm{Alt}(n))$ if $n = 5$
$3 \notin \pi$ $5 \notin \pi$	$\mathrm{Syl}_2(\mathrm{Alt}(n))$	$\mathrm{Syl}_2(\mathrm{Alt}(n))$

Thus the table above implies Theorem D.3 in the case $2 \in \pi$. Assuming $2 \notin \pi$, it follows by [1, D.1, D.2] that if

$$A \in NI(\pi, \mathrm{Sym}(n)) = NI(\pi, \mathrm{Alt}(n)) \qquad (A \in \mathscr{A}(\pi, \infty, \mathrm{Sym}(n))),$$

then it has the following form: It is the π-Hall subgroup of a maximal nilpotent subgroup of $\mathrm{Sym}(n)$, which corresponds to a uniquely defined partition $\{m, n_1, \cdots, n_s\}$ of n satisfying the following conditions:

(a) $m < \min\{p \mid p \in \pi\}$ or $m = 0$.

(b) For each i, $1 \leqq i \leqq s$, $n_i = p_i^{\alpha_i}$ where $p_i \in \pi$ not necessarily distinct and $\alpha_i \geqq 1$.

As $\mathrm{Alt}(n)$ is transitive on the set of all partitions of $\{1, 2, \cdots, n\}$ of the form

$\{m, n_1, \cdots, n_s\}$ and as $\mathrm{Syl}_{p_i}(\mathrm{Sym}(n_i))$ forms a conjugacy class in $\mathrm{Alt}(n_i)$, $N\mathrm{I}(\infty, \mathrm{Alt}(n))$ ($\mathscr{A}(\pi, \infty, \mathrm{Alt}(n))$) consists of a unique conjugacy class in $\mathrm{Alt}(n)$ and Theorem D.3 is proved.

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